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Open string states around a classical solution in vacuum string field theory

Hiroyuki Hata

Department of Physics, Kyoto University
Kyoto 606-8502, Japan
E-mail: hata@gauge.scphys.kyoto-u.ac.jp

Teruhiko Kawano

Department of Physics, University of Tokyo
Hongo, Tokyo 113-0033, Japan
*E-mail:*kawano@hep-th.phys.s.u-tokyo.ac.jp

Abstract: We construct a classical solution of vacuum string field theory (VSFT) and study whether it represents the perturbative open string vacuum. Our solution is given as a squeezed state in the Siegel gauge, and it fixes the arbitrary coefficients in the BRST operator in VSFT. We identify the tachyon and massless vector states as fluctuation modes around the classical solution. The tachyon mass squared $\alpha' m_{\mathrm{t}}^2$ is given in a closed form using the Neumann coefficients defining the three-string vertex, and it reproduces numerically the expected value of -1 to high precision. The ratio of the potential height of the solution to the D25-brane tension is also given in terms of the Neumann coefficients. However, the behavior of the potential height in level truncation does not match our expectation, though there are subtle points in the analysis.

Keywords: Solitons Monopoles and Instantons, Bosonic Strings, D-branes.

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1. Introduction

String field theory has proved to be a powerful tool in understanding the conjectures about tachyon condensation in bosonic open string theory [1, 2, 3]. Among the three conjectures, the potential height problem and the descent relation have been fully understood (see [4] and the references therein), and the current interest has now been focused on the third problem: Is the pure closed string theory without physical open string excitations realized at the tachyon vacuum?

There have appeared a number of works [5, 6, 7] which studied this problem in the level truncation approximation in cubic open string field theory (CSFT) and obtained results supporting the absence of open string excitations at the tachyon vacuum. However, in contrast to the case of the potential height problem [8, 9], the level truncation cannot give a conclusive answer to the third problem since we have to deal with the space of open string states with level number extending to infinity. For a complete understanding of the third problem, we would need the exact solution of tachyon vacuum in CSFT.

Vacuum string field theory (VSFT) [10]–[13] has been proposed to study the third problem in a reverse way. It is an open string field theory which contains no physical open string excitations at all and hence is expected to describe the tachyon vacuum of ordinary CSFT. The action of VSFT is the same as that of CSFT except that the BRST operator Q_{B} is replaced with another one $\mathcal{Q}$ consisting solely of the ghost coordinate. Due to this pure ghost structure of the new BRST operator $\mathcal{Q}$, physical open string spectrum of VSFT becomes trivial.

However, for VSFT to solve the third problem, we have to show that it is connected with the perturbative open string theory.[1] Namely, we have to show that VSFT has a classical solution (let us denote it Ψ_{c}) describing the perturbative open string vacuum. Concretely, Ψ_{c} has to satisfy the following two: First, the physical fluctuation spectrum around Ψ_{c} must reproduce that of the perturbative open string theory. Second, the raise of the energy density of the Ψ_{c} state from that of the trivial vacuum in VSFT must be equal to the D25-brane tension. Elaborated reformulations of VSFT developed recently [14]–[18] would be useful for studying these problems.

The purpose of the present paper is to construct an exact classical solution of VSFT and study whether this solution satisfies the above two requirements. By taking the Siegel gauge, the classical solution has to satisfy two equations, one is the part of $\mathcal{Q}\Psi_{\mathrm{c}} + \Psi_{\mathrm{c}} * \Psi_{\mathrm{c}} = 0$ not proportional to the anti-ghost zero-mode b_0 and the other is the part proportional to b_0 (the latter equation was called the BRST invariance condition in [19]). The former equation is easily solved by assuming the squeezed state form for both the matter and the ghost part of the solution. In fact, such solution including the ghost part has essentially been constructed in [20]. Moreover, it has been claimed [11] that (the matter part of) the solution is identical with the sliver state [21] constructed in a different manner as the matter part of a classical solution in VSFT.[2] On the other hand, Ψ_{c} must also solve the BRST invariance condition. We find that this condition does not impose further constraint on Ψ_{c} but rather it fixes the coefficients in $\mathcal{Q}$ which were arbitrary at the start. Namely, the arbitrary coefficients in $\mathcal{Q}$ are uniquely determined by the requirement

[1] The descent relation has been studied in VSFT to give expected result under the assumption that the ghost parts of the classical solutions for Dp-branes with different p are common [10, 11].

[2] An earlier attempt to construct VSFT solutions by considering ghost number one excitations on the sliver state has appeared in [22].

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that VSFT has a translationally and Lorentz invariant solution in the Siegel gauge.

To test whether our classical solution Ψ_{c} corresponds to the perturbative open string vacuum, we first study the fluctuation spectrum around Ψ_{c}. Namely, we consider the wave equation $\mathcal{Q}_{\mathrm{B}}\Phi = 0$ where $\mathcal{Q}_{\mathrm{B}}$ is the BRST operator for fluctuations around Ψ_{c}. As solutions to the wave equation, we construct a scalar and a vector solution which we expect to represent the tachyon and the massless vector states in perturbative open string theory. For the scalar solution, we obtain from the wave equation $\mathcal{Q}_{\mathrm{B}}\Phi = 0$ an expression of mass squared $\alpha' m_{\mathrm{t}}^2$ of the scalar particle. It is given in a closed form using the Neumann coefficients defining the three-string vertex. Since we cannot evaluate this $\alpha' m_{\mathrm{t}}^2$ analytically at present, we calculate it numerically by using level truncation. The result is just as we expected for the tachyon state: $\alpha' m_{\mathrm{t}}^2$ approaches to -1 to high precision as the level cutoff is increased.

For the vector solution, our analysis is not complete. Though we can show that the solution represents an exactly massless state, the transversality condition for the polarization vector is not imposed by the wave equation. Therefore, more detailed analysis of the solution space including the ghost mode is necessary for the massless sector. As a test of our vector solution we also calculate the tachyon-tachyon-vector coupling to find that it agrees with a familiar one having gauge invariance.

After establishing the tachyon wave function, we proceed to the test of the potential height. For this purpose, we have to identify the D25-brane tension T_{25} and hence the open string coupling constant g_{o}. For calculating g_{o}, we first determine the normalization of the tachyon wave function so that the VSFT action reproduces the canonical kinetic term for the tachyon field. Then, g_{o} is given as the three tachyon coupling on the mass-shell. After these preparations, we obtain the ratio of the energy density $\mathcal{E}_{\mathrm{c}}$ of the Ψ_{c} state to the tension T_{25}. It is given in a closed form in terms of the Neumann coefficients. In particular, the determinant factor appearing in the energy density $\mathcal{E}_{\mathrm{c}}$ is cancelled with that in T_{25}. We calculate the ratio $\mathcal{E}_{\mathrm{c}}/T_{25}$ numerically using level truncation. However, the result is an unwelcome one: $\mathcal{E}_{\mathrm{c}}/T_{25}$ becomes far beyond the expected value of one as the level number cutoff is increased. Though this result is apparently disappointing, we find that there are ambiguities in the analysis of $\mathcal{E}_{\mathrm{c}}/T_{25}$ which need to be fixed before getting a conclusive answer to the potential height problem.

The organization of the rest of this paper is as follows. In section 2, we summarize the VSFT action and various properties of the Neumann coefficients used in later sections. In section 3, the classical solution Ψ_{c} of VSFT is constructed and the coefficients in the BRST operator of VSFT is fixed. In section 4, we construct the tachyon and massless vector wave functions as fluctuation modes around Ψ_{c}, and analyze the tachyon mass. In section 5, we study the potential height of our classical solution. In the final section, we conclude the paper and discuss open questions. In the appendix A, B and C we present some useful formulas and technical details of the calculations.

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2. VSFT action

We shall consider the VSFT described by the following action [10, 11, 13]:

$$\begin{aligned}\mathcal{S}[\Psi] &= -K\left(\frac{1}{2}\Psi\cdot\mathcal{Q}\,\Psi+\frac{1}{3}\Psi\cdot(\Psi*\Psi)\right)\\ &= -K\left(\frac{1}{2}\int_{b_0,x}\langle\Psi|\mathcal{Q}|\Psi\rangle+\frac{1}{3}\int_{b_0,x}^{(3)}\int_{b_0,x}^{(2)}\int_{b_0,x}^{(1)}{}_1\langle\Psi|_2\langle\Psi|_3\langle\Psi|V\rangle_{123}\right). \end{aligned}\tag{2.1}$$

Here, we are taking the representation of diagonalizing the anti-ghost zero-mode b_0, and $\int_{b_0,x}^{(r)}\equiv\int db_0^{(r)}\int d^{26}x_r$ denotes the integration over b_0 and the center-of-mass coordinate x^μ of the r-th string. The string field $|\Psi(x,b_0)\rangle$ is a state in the Fock space of the first quantized string and carries ghost number -1. The three-string vertex $|V\rangle$ is the same as in the ordinary CSFT and is given in the momentum representation for the center-of-mass x^μ as [23]–[28]

$$\begin{aligned}|V\rangle_{123} = \exp\left\{\sum_{r,s=1}^{3}\left(-\sum_{n,m\geq 0}\frac{1}{2}a_n^{(r)\dagger}V_{nm}^{rs}a_m^{(s)\dagger}+\sum_{n\geq 1,m\geq 0}c_n^{(r)\dagger}\widetilde{V}_{nm}^{rs}b_m^{(s)\dagger}\right)\right\}|0\rangle_{123}\times\\ \times(2\pi)^{26}\delta^{26}\left(p_1+p_2+p_3\right),\end{aligned}\tag{2.2}$$

where V_{nm}^{rs} and $\widetilde{V}_{nm}^{rs}$ are the Neumann coefficients for the symmetric three-string connection.[3] The matter (ghost) oscillators satisfy the following (anti-)commutation relations

$$[a_n^{(r)\mu},a_m^{(s)\nu\dagger}]=\eta^{\mu\nu}\delta_{nm}\delta^{rs}\,,\quad \{c_n^{(r)},b_m^{(s)\dagger}\}=\{b_n^{(r)},c_m^{(s)\dagger}\}=\delta_{nm}\delta^{rs}\,,\quad (n,m\geq 1)\,,\tag{2.3}$$

and the oscillator vacuum $|0\rangle$ is defined by $(a_n,b_n,c_n)|0\rangle=0$ $(n\geq 1)$. As for the zero-modes, we have $b_0^{\dagger(r)}=b_0^{(r)}$, $c_0^{\dagger(r)}=c_0^{(r)}=\partial/\partial b_0^{(r)}$ and $a_0^{(r)}=a_0^{\dagger(r)}=\sqrt{2}\,p_r$ ($p_r=-i\partial/\partial x_r$ is the center-of-mass momentum of the string r, and we are adopting the convention of $\alpha'=1$).

The essential difference of the VSFT action from that of the ordinary CSFT is the BRST operator $\mathcal{Q}$ around the tachyon vacuum. It consists purely of ghost operators:

$$\mathcal{Q}=c_0+\sum_{n\geq 1}f_n\mathcal{C}_n\,,\quad \mathcal{C}_n\equiv c_n+(-1)^n c_n^\dagger\,,\tag{2.4}$$

where the coefficient f_n is real (pure imaginary) for an even (odd) n due to the requirement that $\mathcal{Q}$ be hermitian, but otherwise arbitrary at the present stage. Namely,

[3]These Neumann coefficients are related to the ones, N_{nm}^{rs} and X_{nm}^{rs}, in refs. [29, 19] by $V_{nm}^{rs}=-\sqrt{n}N_{nm}^{rs}\sqrt{m}$ $(n,m\geq 1)$, $V_{n0}^{rs}=-\sqrt{n}N_{n0}^{rs}$ $(n\geq 1)$, $V_{00}^{rs}=-N_{00}^{rs}$, and $\widetilde{V}_{nm}^{rs}=-X_{nm}^{rs}$ $(n\geq 1,m\geq 0)$.

$\mathcal{Q}$ satisfies the nilpotency and the Leibniz rule on the $*$-product for arbitrary f_n, and the VSFT action (2.1) has an invariance under the gauge transformation

$$\delta_\Lambda \Psi = \mathcal{Q}\Lambda + \Psi * \Lambda - \Lambda * \Psi\,. \tag{2.5}$$

Since the cohomology of $\mathcal{Q}$ is trivial, the quadratic term $-K(1/2)\Psi{\cdot}\mathcal{Q}\Psi$ of the VSFT action (2.1) supports no physical open string excitations at all.

Here, we shall mention the hermiticity constraint on the string field. The string field Ψ is restricted to satisfy the following hermiticity condition:

$$ {}_2\langle\Psi| = \int_{b_0,x}^{(1)} {}_{12}\langle R|\Psi\rangle_1\,, \tag{2.6}$$

where ${}_{12}\langle R|$ is the reflector given in the momentum representation as

$$\begin{aligned} {}_{12}\langle R| = {}_{12}\langle 0|\exp\left\{-\sum_{n\geq 1}(-1)^n\left(a_n^{(1)}a_n^{(2)} + c_n^{(1)}b_n^{(2)} + c_n^{(2)}b_n^{(1)}\right)\right\}\times \\ \times(2\pi)^{26}\delta^{26}(p_1+p_2)\,\delta(b_0^{(1)}-b_0^{(2)})\,. \end{aligned} \tag{2.7}$$

This constraint reduces the number of degrees of freedom in Ψ to half and ensures the hermiticity of the action (2.1).

In the rest of this section, we shall summarize various useful properties of the Neumann coefficients appearing in the vertex (2.2). Due to the cyclic symmetry property for the three strings, the Neumann coefficients have only three independent components with respect the the upper indices, and we define

$$(V_0)_{nm} = V^{rr}_{nm}\,, \qquad (V_\pm)_{nm} = V^{r,r\pm1}_{nm}\,, \tag{2.8}$$

for $n, m \geq 1$. Then, the twist transformation property of the vertex,

$$\Omega_1\Omega_2\Omega_3|V\rangle_{123} = |V\rangle_{321}\,, \tag{2.9}$$

is translated to the following for the Neumann coefficients:

$$CV_0C = V_0, \qquad CV_\pm C = V_\mp\,. \tag{2.10}$$

Here, Ω_r is the twist operator on the Fock space of the string r:

$$\Omega\left(a_n, b_n, c_n\right)\Omega^{-1} = (-1)^n\left(a_n, b_n, c_n\right)\,, \quad \Omega|0\rangle = |0\rangle\,, \tag{2.11}$$

and C is the twist matrix C defined by

$$C_{nm} = (-1)^n\delta_{nm}\,, \qquad (n, m\geq 1)\,. \tag{2.12}$$

Next, let us define the matrices M_0, M_+ and M_- by

$$M_0 = CV_0, \qquad M_\pm = CV_\pm\,. \tag{2.13}$$

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They enjoy the following two basic properties:
(i) M_α ($\alpha = 0, \pm$) are commutative to each other:

$$[M_0, M_\pm] = [M_+, M_-] = 0\,. \tag{2.14}$$

(ii) M_α satisfy the two identities:

$$M_0 + M_+ + M_- = 1\,, \tag{2.15}$$
$$M_+ M_- = M_0^2 - M_0\,. \tag{2.16}$$

The following formulas are consequences of (2.15) and (2.16):

$$M_0^2 + M_+^2 + M_-^2 = 1\,, \tag{2.17}$$
$$M_+^3 + M_-^3 = 1 - 3M_0^2 + 2M_0^3\,, \tag{2.18}$$
$$M_\pm^2 - M_\pm = M_0 M_\mp\,. \tag{2.19}$$

We also have the corresponding equations for the ghost Neumann coefficients: eqs. (2.8)–(2.19) with V and M replaced by $\widetilde{V}$ and $\widetilde{M}$, respectively.

Next are the formulas concerning the Neumann coefficients V^{rs}_{n0}. Let us define the vectors $\boldsymbol{v}_0$ and $\boldsymbol{v}_\pm$ by

$$(\boldsymbol{v}_0)_n = V^{rr}_{n0}\,, \qquad (\boldsymbol{v}_\pm)_n = V^{r,r\pm 1}_{n0}\,, \qquad (n \geq 1). \tag{2.20}$$

Under the twist transformation, we have

$$C\boldsymbol{v}_0 = \boldsymbol{v}_0\,, \qquad C\boldsymbol{v}_\pm = \boldsymbol{v}_\mp\,. \tag{2.21}$$

Now, the following equations have been known to hold [26, 27]:

$$\sum_{t=1}^{3} \sum_{n\geq 1} V^{rt}_{mn} V^{ts}_{n0} = V^{rs}_{m0}\,, \qquad (m \geq 1) \tag{2.22}$$
$$\sum_{t=1}^{3} \sum_{n\geq 1} V^{tr}_{n0} V^{ts}_{n0} = 2\, V^{rs}_{00}\,. \tag{2.23}$$

Here, we must bear in mind that these equations are valid only when the upper open indices associated with the zero-mode (s in (2.22) and r and s in (2.23)) are contracted with conserved quantities. Using this fact and (2.15), eq. (2.22) is reexpressed as

$$M_+ \boldsymbol{v}_{-0} + M_- \boldsymbol{v}_{+0} = 0\,, \tag{2.24}$$
$$M_0 \boldsymbol{v}_{\pm 0} + (M_\mp - 1)\boldsymbol{v}_{\mp 0} = 0\,, \tag{2.25}$$

where we have introduced abbreviated notations:

$$\boldsymbol{v}_{\pm 0} = \boldsymbol{v}_\pm - \boldsymbol{v}_0\,. \tag{2.26}$$

On the other hand, (2.23) together with

$$V_{00}^{rs} = V_{00}\,\delta^{rs}\,, \qquad V_{00} = \frac{1}{2}\ln\left(\frac{3^3}{2^4}\right), \tag{2.27}$$

and (2.21) gives

$$2\,(\boldsymbol{v}_{+0})^2 - \boldsymbol{v}_{+0}\cdot\boldsymbol{v}_{-0} = 2\,V_{00}\,. \tag{2.28}$$

We do not have the corresponding equations to (2.24), (2.25) and (2.28) for the ghost Neumann coefficients.

3. Classical solution in the Siegel gauge

We would like to construct a (translationally and Lorentz invariant) classical solution Ψ_{c} to the equation of motion of the VSFT action (2.1), $\mathcal{Q}\Psi_{\mathrm{c}} + \Psi_{\mathrm{c}}*\Psi_{\mathrm{c}} = 0$, which is expressed in the Fock space representation as

$$\mathcal{Q}|\Psi_{\mathrm{c}}\rangle_3 + \int_{b_0,x}^{(2)}\int_{b_0,x}^{(1)} {}_1\langle\Psi_{\mathrm{c}}|_2\langle\Psi_{\mathrm{c}}|V\rangle_{123} = 0\,. \tag{3.1}$$

It is the task of later sections to examine whether this solution represents the perturbative open string vacuum.

Due to the pure ghost form of $\mathcal{Q}$, the equation of motion (3.1) can be solved explicitly by adopting the Siegel gauge for the solution, $b_0\Psi_{\mathrm{c}} = 0$. Substituting the expression

$$|\Psi_{\mathrm{c}}\rangle = b_0|\phi_{\mathrm{c}}\rangle\,, \tag{3.2}$$

into (3.1), we find that it consists of the following two equations for $|\phi_{\mathrm{c}}\rangle$:

$$|\phi_{\mathrm{c}}\rangle_3 + {}_1\langle\phi_{\mathrm{c}}|_2\langle\phi_{\mathrm{c}}|\widehat{V}\rangle_{123}\Big|_{p_r=0} = 0\,, \tag{3.3}$$

$$\sum_{n\geq 1} f_n\mathcal{C}_n|\phi_{\mathrm{c}}\rangle_3 + {}_1\langle\phi_{\mathrm{c}}|_2\langle\phi_{\mathrm{c}}|\sum_{r=1}^{3}\sum_{n\geq 1} c_n^{\dagger(r)}\widetilde{V}_{n0}^{r3}|\widehat{V}\rangle_{123}\Big|_{p_r=0} = 0\,, \tag{3.4}$$

where $|\widehat{V}\rangle$ is the reduced vertex without b_0:

$$|\widehat{V}\rangle_{123} = \exp\left\{\sum_{r,s=1}^{3}\left(-\sum_{n,m\geq 0}\frac{1}{2}a_n^{(r)\dagger}V_{nm}^{rs}a_m^{(s)\dagger} + \sum_{n,m\geq 1} c_n^{(r)\dagger}\widetilde{V}_{nm}^{rs}b_m^{(s)\dagger}\right)\right\}|0\rangle_{123}\,. \tag{3.5}$$

The original equation of motion (3.1) is given by eq. (3.3) $-\, b_0^{(3)} \times$ eq. (3.4). If we start with the gauge-fixed action, (2.1) with (3.2) substituted, (3.3) is its equation of motion, while (3.4) is the BRST invariance condition [19].

Let us consider the first equation (3.3). It has been known [20, 11] that (3.3) can be solved (including the ghost part) by assuming the squeezed state form for $|\phi_c\rangle$:[4]

$$|\phi_c\rangle = \mathcal{N}_c \exp\left(-\frac{1}{2}\sum_{n,m\geq 1} a_n^\dagger S_{nm} a_m^\dagger + \sum_{n,m\geq 1} c_n^\dagger \widetilde{S}_{nm} b_m^\dagger\right)|0\rangle\,, \tag{3.6}$$

where S_{nm} and $\widetilde{S}_{nm}$ are unknown real coefficients[5] and $\mathcal{N}_c$ is the (real) normalization factor. We assume further that the state $|\phi_c\rangle$ is twist invariant, $\Omega|\phi_c\rangle = |\phi_c\rangle$, and hence S_{nm} and $\widetilde{S}_{nm}$ satisfy the matrix equations

$$CSC = S\,, \qquad C\widetilde{S}C = \widetilde{S}\,. \tag{3.7}$$

Formulas (A.2) and (A.3) in appendix A imply that, for the squeezed state $|\phi_c\rangle$ (3.6), the second term of (3.3) is again a squeezed state, and moreover it becomes proportional to $|\phi_c\rangle$,

$${}_1\langle\phi_c|_2\langle\phi_c|\widehat{V}\rangle_{123}\Big|_{p_r=0} = \mathcal{N}_c\left[\det(1-S\mathcal{V})\right]^{-13}\det(1-\widetilde{S}\widetilde{\mathcal{V}})\,|\phi_c\rangle_3\,, \tag{3.8}$$

provided S and $\widetilde{S}$ satisfy

$$S = V_0 + (V_+, V_-)(1-S\mathcal{V})^{-1}S\begin{pmatrix} V_- \\ V_+\end{pmatrix}\,, \tag{3.9}$$

and the same one with all the matrices replaced with the tilded ones, respectively. In (3.9), $\mathcal{V}$ is

$$\mathcal{V} = \begin{pmatrix} V_0 & V_+ \\ V_- & V_0\end{pmatrix}\,, \tag{3.10}$$

and S on the RHS should read $\mathrm{diag}(S,S)$. If (3.9) and the corresponding one for $\widetilde{S}$ hold, then $|\phi_c\rangle$ (3.6) becomes a solution to (3.3) by taking the following normalization factor $\mathcal{N}_c$:

$$\mathcal{N}_c = -\left[\det(1-S\mathcal{V})\right]^{13}\left[\det(1-\widetilde{S}\widetilde{\mathcal{V}})\right]^{-1}\,. \tag{3.11}$$

Eq. (3.9) for S has been solved in [20, 11], and we shall summarize the points in obtaining the solution. Defining

$$T = CS = SC\,, \tag{3.12}$$

[4]However, our explicit formulas for the ghost part differ from those in [20].

[5]The hermiticity constraint (2.6) for Ψ_c implies that $\mathcal{O}^*_{nm} = (-1)^{n+m}\mathcal{O}_{nm}$ for both $\mathcal{O} = S$ and $\widetilde{S}$. Imposing further the twist invariance condition (3.7), S_{nm} and $\widetilde{S}_{nm}$ are restricted to be real.

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eq. (3.9) multiplied by C on the left reads

$$T = M_0 + (M_+, M_-)(1 - T\mathcal{M})^{-1} T \begin{pmatrix} M_- \\ M_+ \end{pmatrix} , \tag{3.13}$$

with

$$\mathcal{M} = \begin{pmatrix} M_0 & M_+ \\ M_- & M_0 \end{pmatrix} . \tag{3.14}$$

Let us assume that T commutes with the matrices M_α:

$$[T, M_\alpha] = 0, \qquad (\alpha = 0, \pm) . \tag{3.15}$$

Then, all the matrices in (3.13) are commutative (recall (2.14)), which makes (3.13) fairly easy to deal with. Using the formulas (2.15)–(2.18) for M_α and, in particular,

$$(1 - S\mathcal{V})^{-1} = (1 - T\mathcal{M})^{-1} = \left(1 - 2M_0 T + M_0 T^2\right)^{-1} \begin{pmatrix} 1 - TM_0 & TM_+ \\ TM_- & 1 - TM_0 \end{pmatrix} , \tag{3.16}$$

eq. (3.13) is reduced to

$$(T - 1)\Bigl(M_0 T^2 - (1 + M_0)T + M_0\Bigr) = 0 . \tag{3.17}$$

We do not adopt the solution $T = 1$ which corresponds to the identity state, and take a solution to

$$M_0 T^2 - (1 + M_0)T + M_0 = 0 . \tag{3.18}$$

Among (infinitely) many solutions to (3.18) we take the following one,

$$T = \frac{1}{2M_0}\left(1 + M_0 - \sqrt{(1 - M_0)(1 + 3M_0)}\right) , \tag{3.19}$$

where the branch of the matrix square root is defined by the Taylor expansion,

$$\sqrt{(1 - M_0)(1 + 3M_0)} = 1 + \sum_{k=1}^{\infty} \binom{1/2}{k} \left(2M_0 - 3M_0^2\right)^k , \tag{3.20}$$

and hence T has an expansion in positive powers of M_0: $T = M_0 - M_0^2 + 2M_0^3 + \cdots$. The commutativity (3.15) is evidently satisfied. It has been claimed by numerical comparison that the matrix $S = CT$ with T given by (3.19) is identical to the matrix defining the matter part of the sliver state [11]. However, we do not use this fact explicitly in the rest of this paper.

Determination of the ghost part matrix $\widetilde{S}$ in (3.6) is exactly the same as for S. We take as $\widetilde{T} = C\widetilde{S} = \widetilde{S}C$ the same one (3.19) with M_0 replaced with $\widetilde{M}_0$:

$$\widetilde{T} = \frac{1}{2\widetilde{M}_0}\left(1 + \widetilde{M}_0 - \sqrt{(1-\widetilde{M}_0)(1+3\widetilde{M}_0)}\right). \tag{3.21}$$

Having solved the equation of motion (3.3) including the ghost part, our next task is to consider the BRST invariance condition (3.4). As explained in the introduction, this condition fixes the coefficients f_n in the BRST operator $\mathcal{Q}$ (2.4) of VSFT rather than gives further constraint on the solution $|\phi_c\rangle$.

Using the formula obtained by differentiating (A.3) with respect to ζ_i, the second term of (3.4) is reduced to the form of $|\phi_c\rangle$ operated by $c_n^\dagger$:

$$\begin{aligned}{}_1\langle\phi_c|_2\langle\phi_c|\sum_{r=1}^{3}\sum_{n\geq 1} c_n^{\dagger(r)}\widetilde{V}_{n0}^{r3}|\widehat{V}\rangle_{123}\Big|_{p_r=0} &= \mathcal{N}_c\left[\det(1-S\mathcal{V})\right]^{-13}\det(1-\widetilde{S}\widetilde{\mathcal{V}})\times \\ &\times\sum_{n\geq 1}\left[\widetilde{\boldsymbol{v}}_0 + (\widetilde{V}_+,\widetilde{V}_-)(1-\widetilde{S}\widetilde{\mathcal{V}})^{-1}\widetilde{S}\begin{pmatrix}\widetilde{\boldsymbol{v}}_-\\ \widetilde{\boldsymbol{v}}_+\end{pmatrix}\right]_n c_n^{\dagger(3)}|\phi_c\rangle_3,\end{aligned} \tag{3.22}$$

where the vectors $\widetilde{\boldsymbol{v}}_\alpha$ are defined by

$$\left(\widetilde{\boldsymbol{v}}_0\right)_n = \widetilde{V}_{n0}^{r,r}, \qquad \left(\widetilde{\boldsymbol{v}}_\pm\right)_n = \widetilde{V}_{n0}^{r,r\pm 1}. \tag{3.23}$$

Then, using (3.11) and

$$\mathcal{C}_n|\phi_c\rangle = \sum_{m\geq 1} c_m^\dagger\left(C_{mn} - \widetilde{S}_{mn}\right)|\phi_c\rangle, \tag{3.24}$$

the BRST invariance condition (3.4) holds if f_n ($n \geq 1$) are given by

$$\begin{aligned}\boldsymbol{f} &= (C-\widetilde{S})^{-1}\left[\widetilde{\boldsymbol{v}}_0 + (\widetilde{V}_+,\widetilde{V}_-)(1-\widetilde{S}\widetilde{\mathcal{V}})^{-1}\widetilde{S}\begin{pmatrix}\widetilde{\boldsymbol{v}}_-\\ \widetilde{\boldsymbol{v}}_+\end{pmatrix}\right] \\ &= (1-\widetilde{T})^{-1}\left[\widetilde{\boldsymbol{v}}_0 + (\widetilde{M}_+,\widetilde{M}_-)(1-\widetilde{T}\widetilde{\mathcal{M}})^{-1}\widetilde{T}\begin{pmatrix}\widetilde{\boldsymbol{v}}_+\\ \widetilde{\boldsymbol{v}}_-\end{pmatrix}\right],\end{aligned} \tag{3.25}$$

where we have used that

$$C\widetilde{\boldsymbol{v}}_0 = \widetilde{\boldsymbol{v}}_0, \quad C\widetilde{\boldsymbol{v}}_\pm = \widetilde{\boldsymbol{v}}_\mp. \tag{3.26}$$

The vector $\boldsymbol{f}$ obtained this way satisfies $C\boldsymbol{f} = \boldsymbol{f}$, namely, $f_{2n+1} = 0$.

As explained in [10], a homogeneous field redefinition $\Psi \to \exp\left(\sum_{n\geq 1}\epsilon_n K_n\right)\Psi$ with $K_n = L_n - (-1)^n L_{-n}$ maps VSFT into another VSFT having $\mathcal{Q}$ with different coefficients f_n. Therefore, $\exp\left(\sum_{n\geq 1}\epsilon_n K_n\right)\Psi_c$ is a solution to the equation of motion in VSFT with f_n different from (3.25). However, this field redefinition takes our classical solution Ψ_c away from the Siegel gauge.

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4. Fluctuation spectrum around Ψ_c

Since we have obtained a classical solution Ψ_c in VSFT, let us next examine whether Ψ_c represents the perturbative open string vacuum. Expanding the original string field Ψ in VSFT as

$$\Psi = \Psi_c + \Phi\,, \tag{4.1}$$

with Φ being the fluctuation, the VSFT action (2.1) is expressed as

$$\mathcal{S}[\Psi] = \mathcal{S}[\Psi_c] - K\left(\frac{1}{2}\Phi\cdot\mathcal{Q}_B\Phi + \frac{1}{3}\Phi\cdot(\Phi*\Phi)\right), \tag{4.2}$$

where $\mathcal{Q}_B$ is defined by

$$\mathcal{Q}_B\Phi = \mathcal{Q}\,\Phi + \Psi_c*\Phi + \Phi*\Psi_c\,. \tag{4.3}$$

The new BRST operator $\mathcal{Q}_B$ satisfies the nilpotency and the Leibniz rule on the $*$-product.

What we have to test for confirming that Ψ_c represents the perturbative open string vacuum are the following two:

- Whether the quadratic term of the fluctuation, $(1/2)\Psi\cdot\mathcal{Q}_B\Psi$, supports the known perturbative open string spectrum.

- Whether $\mathcal{S}[\Psi_c]$ has the expected value of the D25-brane tension:

$$-\mathcal{S}[\Psi_c] = T_{25}V_{26}\,, \tag{4.4}$$

  where $V_{26} = \int d^{26}x$ is the space-time volume.

The test of (4.4) needs the expression of T_{25} and hence that of the open string coupling constant g_o in terms of the parameters in VSFT. Since g_o is defined by the three-tachyon on-shell amplitude, what we have to examine first of all is the fluctuation spectrum, namely, whether $(1/2)\Psi\cdot\mathcal{Q}_B\Psi$ really contains tachyon and photon etc.

4.1 Tachyon wave function

We shall construct the tachyon wave function Φ_t which is a scalar solution to

$$\mathcal{Q}_B\Phi_t = 0\,, \tag{4.5}$$

and carries center-of-mass momentum $p^2 = 1$ (recall that we are adopting the convention $\alpha' = 1$). Of course, Φ_t must not be $\mathcal{Q}_B$-exact. Since Φ_t should be twist invariant, $\Omega\Phi_t = \Phi_t$, (4.5) is rewritten as

$$\mathcal{Q}\,\Phi_t + (1+\Omega)\,(\Psi_c*\Phi_t) = 0\,. \tag{4.6}$$

Let us take the Siegel gauge for $\Phi_{\rm t}$:

$$|\Phi_{\rm t}\rangle = b_0|\phi_{\rm t}\rangle\,. \tag{4.7}$$

Then, the wave equation (4.6) consists of the following two:

$$|\phi_{\rm t}\rangle_3 + (1+\Omega_3)\,{}_1\langle\phi_{\rm c}|_2\langle\phi_{\rm t}|\widehat{V}\rangle_{123}\Big|_{p_1=0,\,p_2=-p_3} = 0, \tag{4.8}$$

$$\sum_{n\geq1} f_n\mathcal{C}_n|\phi_{\rm t}\rangle_3 + (1+\Omega_3)\,{}_1\langle\phi_{\rm c}|_2\langle\phi_{\rm t}|\sum_{r=1}^{3}\sum_{n\geq1}c_n^{\dagger(r)}\widetilde{V}_{n0}^{r3}|\widehat{V}\rangle_{123}\Big|_{p_1=0,\,p_2=-p_3} = 0\,. \tag{4.9}$$

Now, let us try the following $|\phi_{\rm t}\rangle$ obtained as a simple modification on $|\phi_{\rm c}\rangle$ (3.6):

$$\begin{aligned}|\phi_{\rm t}\rangle &= \frac{\mathcal{N}_{\rm t}}{\mathcal{N}_{\rm c}}\exp\left(-\sum_{n\geq1}t_n a_n^\dagger a_0\right)|\phi_{\rm c}\rangle\\ &= \mathcal{N}_{\rm t}\exp\left(-\frac{1}{2}\sum_{n,m\geq1}a_n^\dagger S_{nm}a_m^\dagger + \sum_{n,m\geq1}c_n^\dagger\widetilde{S}_{nm}b_m^\dagger - \sum_{n\geq1}t_n a_n^\dagger a_0\right)|0\rangle\,. \end{aligned}\tag{4.10}$$

In particular, the ghost part of $|\phi_{\rm t}\rangle$ is the same as that of $|\phi_{\rm c}\rangle$. Though not written explicitly, $|\phi_{\rm t}\rangle$ carries non-vanishing momentum in contrast with $|\phi_{\rm c}\rangle$ which is translationally invariant. Since $|\phi_{\rm t}\rangle$ is twist invariant, the coefficient t_n is non-vanishing only for even n, namely, the vector $\boldsymbol{t}$ satisfies

$$C\boldsymbol{t} = \boldsymbol{t}\,. \tag{4.11}$$

Moreover, $\boldsymbol{t}$ is real due to the hermiticity (2.6). The normalization factor $\mathcal{N}_{\rm t}$ for $|\phi_{\rm t}\rangle$ is not determined by (4.8) and (4.9) which are linear in $|\phi_{\rm t}\rangle$. It is fixed by a condition given in the next section.

The second term of (4.8) for $|\phi_{\rm t}\rangle$ of (4.10) is calculated by using (A.2) and is given by

$${}_1\langle\phi_{\rm c}|_2\langle\phi_{\rm t}|\widehat{V}\rangle_{123}\Big|_{p_1=0,\,p_2=-p_3} = -\frac{\mathcal{N}_{\rm t}}{\mathcal{N}_{\rm c}}\exp\left(-\sum_{n\geq1}u_n a_n^{(3)\dagger}a_0^{(3)} - \frac{1}{2}G\,(a_0^{(3)})^2\right)|\phi_{\rm c}\rangle_3\,, \tag{4.12}$$

where u_n and G on the RHS are

$$\boldsymbol{u} = \boldsymbol{v}_0 - \boldsymbol{v}_- + (V_+,V_-)(1-S\mathcal{V})^{-1}S\begin{pmatrix}\boldsymbol{v}_- - \boldsymbol{v}_+\\ \boldsymbol{v}_+ - \boldsymbol{v}_0\end{pmatrix} + (V_+,V_-)(1-S\mathcal{V})^{-1}\begin{pmatrix}0\\ \boldsymbol{t}\end{pmatrix}, \tag{4.13}$$

$$\begin{aligned}G &= 2\,V_{00} + (\boldsymbol{v}_- - \boldsymbol{v}_+, \boldsymbol{v}_+ - \boldsymbol{v}_0)(1-S\mathcal{V})^{-1}S\begin{pmatrix}\boldsymbol{v}_- - \boldsymbol{v}_+\\ \boldsymbol{v}_+ - \boldsymbol{v}_0\end{pmatrix} +\\ &\quad +2\,(\boldsymbol{v}_- - \boldsymbol{v}_+, \boldsymbol{v}_+ - \boldsymbol{v}_0)(1-S\mathcal{V})^{-1}\begin{pmatrix}0\\ \boldsymbol{t}\end{pmatrix} + (0,\boldsymbol{t})\mathcal{V}(1-S\mathcal{V})^{-1}\begin{pmatrix}0\\ \boldsymbol{t}\end{pmatrix}\,. \end{aligned}\tag{4.14}$$

Eq. (4.12) implies that $|\phi_{\rm t}\rangle$ of (4.10) becomes a solutions to (4.8) if $\boldsymbol{t}$ satisfies

$$\boldsymbol{t} = \boldsymbol{u}\,, \tag{4.15}$$

and the center-of-mass momentum $a_0 = \sqrt{2}\,p$ carried by $|\phi_{\rm t}\rangle$ satisfies

$$\exp\left(-\frac{1}{2}\,G\,(a_0)^2\right) = \frac{1}{2}\,. \tag{4.16}$$

Note that the twist operator Ω_3 in (4.8) is effectively equal to the identity since (4.12) is twist invariant under (4.15). Moreover, in this case, the second equation (4.9) is also satisfied since the ghost part of $|\phi_{\rm t}\rangle$ is the same as that of $|\phi_{\rm c}\rangle$.

Therefore, what we have to do is to first determine the vector $\boldsymbol{t}$ satisfying (4.15) and then to check whether the momentum p satisfying (4.16) really reproduces the tachyon mass, $p^2 = 1$.

4.2 Determination of $\boldsymbol{t}$

Let us solve (4.15) for $\boldsymbol{t}$, namely,

$$\boldsymbol{t} = -\boldsymbol{v}_{+0} + (M_+, M_-)(1 - T\mathcal{M})^{-1}T\begin{pmatrix}\boldsymbol{v}_{+0} - \boldsymbol{v}_{-0}\\ \boldsymbol{v}_{-0}\end{pmatrix} + (M_+, M_-)(1 - T\mathcal{M})^{-1}\begin{pmatrix}0\\ \boldsymbol{t}\end{pmatrix}, \tag{4.17}$$

which is obtained by multiplying (4.15) by C on the left and using (2.21),(2.26) and (4.11). After a tedious calculation using

$$(1 - S\mathcal{V})^{-1} = (1 - T\mathcal{M})^{-1} = \left[(1 - M_0)(1 + T)\right]^{-1}\begin{pmatrix}1 - TM_0 & TM_+\\ TM_- & 1 - TM_0\end{pmatrix}, \tag{4.18}$$

valid for T satisfying (3.18) (recall (3.16)), and the formulas (2.15)–(2.19), (2.24) and (2.25) for M_α and $\boldsymbol{v}_\alpha$, eq. (4.17) multiplied by $(1 - M_0)(1 + T)$ is reduced to the following simple equation:

$$(M_+ + M_-T)\boldsymbol{t} = (T - 1)\left(\boldsymbol{v}_{+0} - T\boldsymbol{v}_{-0}\right). \tag{4.19}$$

Note first that we cannot invert $M_+ + M_-T$ to solve (4.19) since the following identity holds for T satisfying (3.18):[6]

$$(M_- + M_+T)(M_+ + M_-T) = 0\,. \tag{4.20}$$

Therefore, solution to (4.19) is not unique. However, twist invariant solution satisfying (4.11) does exist uniquely as we shall show. A twist invariant solution $\boldsymbol{t}$ must satisfy both (4.19) and the same one multiplied by C:

$$(M_- + M_+T)\boldsymbol{t} = (T - 1)\left(\boldsymbol{v}_{-0} - T\boldsymbol{v}_{+0}\right). \tag{4.21}$$

[6] For consistency, the RHS of (4.19) operated by $M_- + M_+T$ must vanish. This is indeed the case since we have $(M_+ + M_-T)\left(\boldsymbol{v}_{+0} - T\boldsymbol{v}_{-0}\right) = 0$.

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L	G	$-m_t^2$
10	0.6493	1.06746
50	0.6732	1.02958
100	0.6773	1.02333
150	0.6791	1.02072
200	0.6801	1.01918
250	0.6808	1.01812
300	0.6813	1.01734

L	G	$-m_t^2$
9	0.6665	1.04000
49	0.6760	1.02534
99	0.6786	1.02141
149	0.6799	1.01951
199	0.6807	1.01830
249	0.6813	1.01744
299	0.6817	1.01678

Table 1: G and $-m_t^2$ for various values of the cutoff L. The left (right) table shows the result for even (odd) L.

As the solution of the equation obtained by adding (4.19) and (4.21), we get

$$\boldsymbol{t} = -\left[(1-M_0)(1+T)\right]^{-1}(1-T)^2\left(\boldsymbol{v}_{+0}+\boldsymbol{v}_{-0}\right). \tag{4.22}$$

It is a straightforward exercise to show that the solution (4.22) also satisfies the equation obtained by subtracting the two.

4.3 Tachyon mass

Having obtained the vector $\boldsymbol{t}$, we shall proceed to the determination of the mass of the state Φ_t. We shall first calculate G (4.14), which is rewritten as

$$\begin{aligned} G = {} & 2\,V_{00} + (\boldsymbol{v}_{-0}-\boldsymbol{v}_{+0},\boldsymbol{v}_{+0})(1-T\mathcal{M})^{-1}T\begin{pmatrix}\boldsymbol{v}_{+0}-\boldsymbol{v}_{-0}\\ \boldsymbol{v}_{-0}\end{pmatrix} + \\ & +2\,(\boldsymbol{v}_{-0}-\boldsymbol{v}_{+0},\boldsymbol{v}_{+0})(1-T\mathcal{M})^{-1}\begin{pmatrix}0\\ \boldsymbol{t}\end{pmatrix} + (0,\boldsymbol{t})\mathcal{M}(1-T\mathcal{M})^{-1}\begin{pmatrix}0\\ \boldsymbol{t}\end{pmatrix}. \end{aligned} \tag{4.23}$$

and then obtain the mass from (4.16). We present here a concise expression of G with $\boldsymbol{t}$ given by (4.22):

$$G = \boldsymbol{v}_{+0}G_{++}\boldsymbol{v}_{+0} + \boldsymbol{v}_{+0}G_{+-}\boldsymbol{v}_{-0}\,, \tag{4.24}$$

with

$$G_{++} = -2\,M_0\,(1-M_0)^{-1}(1+3M_0)^{-2}\Big((3M_0-1)T + 9M_0^2 - 3\Big)\,, \tag{4.25}$$

$$G_{+-} = (1+M_0)\,(1-M_0)^{-1}(1+3M_0)^{-2}\Big((3M_0-1)T + 9M_0^2 - 3\Big)\,. \tag{4.26}$$

The $2\,V_{00}$ term in (4.23) has been included in (4.24) by use of (2.28). Derivation of (4.24) is rather tedious and is summarized in appendix B.

We have not succeeded in analytically evaluating the value of G. Instead, we have calculated G numerically by level truncation approximation. Namely, we restrict the indices n, m of the matrix M_0 and the vectors $\boldsymbol{v}_{\pm 0}$ to $1 \le n, m \le L$ and calculate G (4.24) for various values of the cutoff L. The matrix T for a finite L is defined by (3.19). Table 1 summarizes the result of our calculation. For each value of L, we calculated G and the corresponding mass squared $m_{\rm t}^2$ obtained from (4.16):

$$e^{G m_{\rm t}^2} = \frac{1}{2}\,. \tag{4.27}$$

As seen from the table, $m_{\rm t}^2$ tends to the expected value of the tachyon mass squared, -1, as we increase the cutoff L (a slightly better result is obtained for odd L than even L). The value of $-m_{\rm t}^2$ at $L = \infty$ predicted by a fitting function of the form $\sum_{k=0}^{3} c_k (\ln L)^{-k}$ is 1.0013 (0.9947) for even (odd) L. Though rigorous and analytic evaluation of $m_{\rm t}^2$ is of course desired, our analysis here strongly supports that the wave function $\Phi_{\rm t}$ really represents the tachyon mode on the perturbative open string vacuum.

4.4 Massless vector mode

In the previous subsections, we have succeeded in identifying the tachyon mode as a fluctuation mode around the VSFT solution $\Psi_{\rm c}$. As another test of the fluctuation spectrum around $\Psi_{\rm c}$, let us try constructing the wave function $\Phi_{\rm v}$ representing the massless vector state on the perturbative vacuum. Here again, we try the Siegel gauge solution

$$|\Phi_{\rm v}\rangle = b_0 |\phi_{\rm v}\rangle\,, \tag{4.28}$$

and consider $|\phi_{\rm v}\rangle$ of the following form:

$$|\phi_{\rm v}\rangle = \left(\sum_{n=1,3,5,\ldots} d_n^{\mu} a_n^{\mu\dagger} \right) |\phi_{\rm t}\rangle\,. \tag{4.29}$$

Since we are interested in the massless states which should be odd under the twist transformation, we assume that vector d_n^{μ} has nonvanishing (and real) components only for odd n:

$$C \boldsymbol{d}^{\mu} = -\boldsymbol{d}^{\mu}\,. \tag{4.30}$$

Then, the wave equation $\mathcal{Q}_{\rm B} \Phi_{\rm v} = 0$ consists of the following two for $\phi_{\rm v}$:

$$|\phi_{\rm v}\rangle_3 + (1 - \Omega_3)\ {}_1\langle \phi_{\rm c}|_2 \langle \phi_{\rm v}|\widehat{V}\rangle_{123}\Big|_{p_1=0,\,p_2=-p_3} = 0\,, \tag{4.31}$$

$$\sum_{n \ge 1} f_n \mathcal{C}_n |\phi_{\rm v}\rangle_3 + (1 - \Omega_3)\ {}_1\langle \phi_{\rm c}|_2 \langle \phi_{\rm v}| \sum_{r=1}^{3} \sum_{n \ge 1} c_n^{\dagger(r)} \widetilde{V}_{n0}^{r3} |\widehat{V}\rangle_{123}\Big|_{p_1=0,\,p_2=-p_3} = 0\,. \tag{4.32}$$

Let us consider the second term of (4.31). Using the formula (A.2) differentiated with respect to K_i, we have

$$
\begin{aligned}
{}_1\langle\phi_{\rm c}|_2\langle\phi_{\rm v}|\widehat{V}\rangle_{123} = -\exp\left(-\frac{1}{2}G(a_0^{(3)})^2\right)\Biggl\{-\sum_{n\ge1}\left[(V_+,V_-)(1-S\mathcal{V})^{-1}\begin{pmatrix}0\\ \boldsymbol{d}^\mu\end{pmatrix}\right]_n a_n^{(3)\mu\dagger} - \\
-\Bigl[(0,\boldsymbol{t})\mathcal{V}+(\boldsymbol{v}_- - \boldsymbol{v}_+,\boldsymbol{v}_+ - \boldsymbol{v}_0)\Bigr]\times \\
\times(1-S\mathcal{V})^{-1}\begin{pmatrix}0\\ \boldsymbol{d}^\mu\end{pmatrix}a_0^{(3)\mu}\Biggr\}|\phi_{\rm t}\rangle_3\,. \qquad (4.33)
\end{aligned}
$$

The $a_0^{(3)\mu}$ term in (4.33) does not survive the twist-odd projection. For the $a_n^{(3)\mu\dagger}$ term operated by $1-\Omega_3$, we use the following identity valid for an arbitrary $\boldsymbol{d}^\mu$ and proved by using (4.18), (2.19) and (4.30):

$$
(1-C)(V_+,V_-)(1-S\mathcal{V})^{-1}\begin{pmatrix}0\\ \boldsymbol{d}^\mu\end{pmatrix} = -\boldsymbol{d}^\mu\,. \qquad (4.34)
$$

Therefore, we have

$$
(1-\Omega_3)\,{}_1\langle\phi_{\rm c}|_2\langle\phi_{\rm v}|\widehat{V}\rangle_{123} = -\exp\left(-\frac{1}{2}G(a_0^{(3)})^2\right)|\phi_{\rm v}\rangle_3\,, \qquad (4.35)
$$

for any $\boldsymbol{d}^\mu$ parameterizing the state $|\phi_{\rm v}\rangle$. Eq. (4.35) implies that the first of the wave equation, (4.31), holds if the center-of-mass momentum carried by $|\phi_{\rm v}\rangle$ satisfies $p^2=0$. Namely, our $|\Phi_{\rm v}\rangle$ represents a massless state for any $\boldsymbol{d}^\mu$.

Once the first equation (4.31) is satisfied, the second one (4.32) holds automatically since the ghost part of ${}_1\langle\phi_{\rm c}|_2\langle\phi_{\rm v}|\sum_{r=1}^3\sum_{n\ge1}c_n^{\dagger(r)}\widetilde{V}_{n0}^{r3}|\widehat{V}\rangle_{123}$ is the same as that for $\phi_{\rm t}$ and is twist invariant. Therefore, the whole wave equation $\mathcal{Q}_{\rm B}\Phi_{\rm v}=0$ is satisfied by the present $\Phi_{\rm v}$ only if we set $p^2=0$. In particular, the vector $\boldsymbol{d}^\mu$ can be completely arbitrary. However, this is an unwelcome fact. We expect that the wave equation $\mathcal{Q}_{\rm B}\Phi_{\rm v}=0$ gives the transversality condition on the polarization vector as well as the on-shell condition as in the case of ordinary BRST operator $Q_{\rm B}=\sum_n c_{-n}\left(L_n^{\rm matt}+(1/2)L_n^{\rm gh}\right)$. In the present case, $\boldsymbol{d}^\mu$ is fixed neither as a space-time vector nor as a vector in the level number space.

We do not know whether this trouble is merely an artifact of taking the Siegel gauge for $\Phi_{\rm v}$ or it indicates that our classical solution $\Psi_{\rm c}$ does not represent the perturbative open string vacuum. A more detailed analysis of the massless fluctuation mode space including the ghost sector is necessary to get a definite answer. Here, as a partial support for our identification of $|\Phi_{\rm v}\rangle$ as the massless vector state, we present the result of our calculation of the tachyon-tachyon-vector coupling ${}_1\langle\phi_{\rm t}|_2\langle\phi_{\rm t}|_3\langle\phi_{\rm v}|\widehat{V}\rangle_{123}$ using the present $|\phi_{\rm v}\rangle$ of (4.29). We find that its momentum dependence takes a familiar form $(p_1-p_2)_\mu\zeta^\mu$ with the "effective polarization vector"

ζ^μ given by[7]

$$\zeta^\mu = \boldsymbol{v}_{+0}\left[(1-M_0)(1+3M_0)^2\right]^{-1} M_0^2(2-T)\,\boldsymbol{d}^\mu\,. \tag{4.36}$$

5. Potential height

In the previous section, we found that the fluctuation around $\Psi_{\rm c}$ contains at least the tachyon and the massless vector modes. In this section, we shall carry out the test of (4.4) concerning the energy density of the solution $\Psi_{\rm c}$. For this purpose, we shall first calculate various quantities: energy density $-\mathcal{S}[\Psi_{\rm c}]/V_{26}$, normalization factor $\mathcal{N}_{\rm t}$ of the tachyon wave function, and the open string coupling constant $g_{\rm o}$. After these preparations, we shall proceed to the examination of (4.4).

5.1 Energy density $\mathcal{E}_{\rm c}$

In this subsection, we shall evaluate the energy density $\mathcal{E}_{\rm c}$ of the solution $\Psi_{\rm c}$. Using the equation of motion for $\Psi_{\rm c}$, $\mathcal{Q}\Psi_{\rm c} + \Psi_{\rm c}*\Psi_{\rm c} = 0$, we have

$$\begin{aligned}\mathcal{S}[\Psi_{\rm c}] &= -\frac{K}{6}\int_{b_0,x}\langle\Psi_{\rm c}|\mathcal{Q}|\Psi_{\rm c}\rangle = -\frac{K}{6}\langle\phi_{\rm c}|\phi_{\rm c}\rangle\,V_{26}\\ &= -\frac{K}{6}\,\mathcal{N}_{\rm c}^2\left[\det(1-S^2)\right]^{-13}\det(1-\widetilde{S}^2)\,V_{26}.\end{aligned} \tag{5.1}$$

When T satisfies (3.18), the normalization factor $\mathcal{N}_{\rm c}$ (3.11) is written as

$$\mathcal{N}_{\rm c} = -\left(\det\left[(1-M_0)(1+T)\right]\right)^{13}\left(\det\left[(1-\widetilde{M}_0)(1+\widetilde{T})\right]\right)^{-1}. \tag{5.2}$$

Then, using (3.19) and (3.21), in particular,

$$\frac{1+T}{1-T} = \sqrt{\frac{1+3M_0}{1-M_0}}\,, \tag{5.3}$$

the energy density of our solution $\Psi_{\rm c}$ is given by

$$\begin{aligned}\mathcal{E}_{\rm c} = -\frac{\mathcal{S}[\Psi_{\rm c}]}{V_{26}} = \frac{K}{6}&\left(\det\left[(1-M_0)^{3/2}(1+3M_0)^{1/2}\right]\right)^{13}\times\\ &\times\left(\det\left[(1-\widetilde{M}_0)^{3/2}(1+3\widetilde{M}_0)^{1/2}\right]\right)^{-1}.\end{aligned} \tag{5.4}$$

5.2 Normalization factor $\mathcal{N}_{\rm t}$

We determine the normalization factor $\mathcal{N}_{\rm t}$ in (4.10) by the requirement that the "tachyon field" $\varphi_{\rm t}(x)$ which appear in the expansion of the fluctuation Φ (4.1) as

$$|\Phi\rangle = |\Phi_{\rm t}\rangle\varphi_{\rm t}(x) + \cdots\,, \tag{5.5}$$

[7] The original expression using the quantities defined in section 5 is $(A_0^\mu \boldsymbol{t}\mathcal{V}_3 - \boldsymbol{V}^\mu)(1 - S\mathcal{V}_3)^{-1}(0,0,\boldsymbol{d}^\mu)^{\rm T}$. The derivation of (4.36) is similar to that of H (5.13) explained in appendix C.

has canonical kinetic term in $-K(1/2)\Phi\cdot\mathcal{Q}_{\mathrm{B}}\Phi$ (see eq. (4.2)). Therefore, let us consider the following quantity for Φ_{t} with off-shell p^2 ($p^2\neq -m_{\mathrm{t}}^2$) (recall eqs. (4.8) and (4.12)):

$$
\begin{aligned}
\int db_0\,\langle\Phi_{\mathrm{t}}|\mathcal{Q}_{\mathrm{B}}|\Phi_{\mathrm{t}}\rangle &= {}_3\langle\phi_{\mathrm{t}}|\left(|\phi_{\mathrm{t}}\rangle_3 + 2\,{}_1\langle\phi_{\mathrm{c}}|_2\langle\phi_{\mathrm{t}}|\widehat{V}\rangle_{123}\right)\\
&= \left(1-2\,e^{-Gp^2}\right)\langle\phi_{\mathrm{t}}|\phi_{\mathrm{t}}\rangle = \left(1-e^{-G(p^2+m_{\mathrm{t}}^2)}\right)\langle\phi_{\mathrm{t}}|\phi_{\mathrm{t}}\rangle\,, \qquad (5.6)
\end{aligned}
$$

where we have used (4.27). The inner product $\langle\phi_{\mathrm{t}}|\phi_{\mathrm{t}}\rangle$ is given by

$$
\langle\phi_{\mathrm{t}}|\phi_{\mathrm{t}}\rangle = \mathcal{N}_{\mathrm{t}}^2\left[\det(1-S^2)\right]^{-13}\det(1-\widetilde{S}^2)\,\exp\!\big(2\,\boldsymbol{t}\,(1+T)^{-1}\boldsymbol{t}\,p^2\big)\,. \qquad (5.7)
$$

Taylor expanding (5.6) around $p^2=-m_{\mathrm{t}}^2$, we have

$$
\begin{aligned}
K\int db_0\,\langle\Phi_{\mathrm{t}}|\mathcal{Q}_{\mathrm{B}}|\Phi_{\mathrm{t}}\rangle \underset{p^2\sim -m_{\mathrm{t}}^2}{\sim} & K\mathcal{N}_{\mathrm{t}}^2 G\left[\det(1-S^2)\right]^{-13}\det(1-\widetilde{S}^2)\times\\
& \exp\!\big(-2\,\boldsymbol{t}\,(1+T)^{-1}\boldsymbol{t}\,m_{\mathrm{t}}^2\big)\cdot\big(p^2+m_{\mathrm{t}}^2\big)\,, \qquad (5.8)
\end{aligned}
$$

from which $\mathcal{N}_{\mathrm{t}}$ is read off as

$$
\mathcal{N}_{\mathrm{t}} = \left(\frac{1}{KG}\right)^{1/2}\left[\det(1-S^2)\right]^{13/2}\left[\det(1-\widetilde{S}^2)\right]^{-1/2}\exp\!\Big(\boldsymbol{t}\,(1+T)^{-1}\boldsymbol{t}\,m_{\mathrm{t}}^2\Big)\,. \qquad (5.9)
$$

5.3 g_{o} as three-tachyon on-shell coupling

The open string coupling constant g_{o} is defined to be the three-tachyon on-shell amplitude. Using the tachyon wave function Φ_{t} with normalized kinetic term, g_{o} is given by

$$
\begin{aligned}
g_{\mathrm{o}} &= K\,{}_1\langle\phi_{\mathrm{t}}|_2\langle\phi_{\mathrm{t}}|_3\langle\phi_{\mathrm{t}}|\widehat{V}\rangle_{123}\Big|_{p_1^2=p_2^2=p_3^2=-m_{\mathrm{t}}^2}\\
&= K\mathcal{N}_{\mathrm{t}}^3\,{}_{123}\langle 0|\exp\left(-\frac{1}{2}\boldsymbol{A}S\boldsymbol{A}-A_0\boldsymbol{t}\boldsymbol{A}-\sum_{r=1}^{3}\sum_{n,m\geq 1}c_n^{(r)}\widetilde{S}_{nm}b_m^{(r)}\right)\times\\
&\quad\times\exp\left(-\frac{1}{2}\boldsymbol{A}^\dagger\mathcal{V}_3\boldsymbol{A}^\dagger-\boldsymbol{A}^\dagger\boldsymbol{V}-\frac{1}{2}V_{00}(A_0)^2+\sum_{r,s=1}^{3}\sum_{n,m\geq 1}c_n^{(r)\dagger}\widetilde{V}^{rs}_{nm}b_m^{(s)\dagger}\right)|0\rangle_{123}\\
&= K\mathcal{N}_{\mathrm{t}}^3\left[\det(1-S\mathcal{V}_3)\right]^{-13}\det(1-\widetilde{S}\widetilde{\mathcal{V}}_3)\,\exp\Bigg\{-\frac{1}{2}\boldsymbol{V}(1-S\mathcal{V}_3)^{-1}S\boldsymbol{V}+\\
&\quad+\boldsymbol{V}(1-S\mathcal{V}_3)^{-1}\boldsymbol{t}A_0-\frac{1}{2}A_0\boldsymbol{t}\mathcal{V}_3(1-S\mathcal{V}_3)^{-1}\boldsymbol{t}A_0-\frac{1}{2}V_{00}(A_0)^2\Bigg\}\,, \qquad (5.10)
\end{aligned}
$$

with various new quantities defined by

$$\boldsymbol{A}=\begin{pmatrix}a_n^{(1)}\\ a_n^{(2)}\\ a_n^{(3)}\end{pmatrix},\quad A_0=\begin{pmatrix}a_0^{(1)}\\ a_0^{(2)}\\ a_0^{(3)}\end{pmatrix},\quad \mathcal{V}_3=\begin{pmatrix}V_0 & V_+ & V_-\\ V_- & V_0 & V_+\\ V_+ & V_- & V_0\end{pmatrix},\quad \widetilde{\mathcal{V}}_3=\begin{pmatrix}\widetilde{V}_0 & \widetilde{V}_+ & \widetilde{V}_-\\ \widetilde{V}_- & \widetilde{V}_0 & \widetilde{V}_+\\ \widetilde{V}_+ & \widetilde{V}_- & \widetilde{V}_0\end{pmatrix},$$

$$\boldsymbol{V}=\begin{pmatrix}\boldsymbol{v}_0 & \boldsymbol{v}_+ & \boldsymbol{v}_-\\ \boldsymbol{v}_- & \boldsymbol{v}_0 & \boldsymbol{v}_+\\ \boldsymbol{v}_+ & \boldsymbol{v}_- & \boldsymbol{v}_0\end{pmatrix}\begin{pmatrix}a_0^{(1)}\\ a_0^{(2)}\\ a_0^{(3)}\end{pmatrix}=\begin{pmatrix}a_0^{(2)}\boldsymbol{v}_{+0}+a_0^{(3)}\boldsymbol{v}_{-0}\\ a_0^{(3)}\boldsymbol{v}_{+0}+a_0^{(1)}\boldsymbol{v}_{-0}\\ a_0^{(1)}\boldsymbol{v}_{+0}+a_0^{(2)}\boldsymbol{v}_{-0}\end{pmatrix}=\begin{pmatrix}a_0^{(2)} & a_0^{(3)}\\ a_0^{(3)} & a_0^{(1)}\\ a_0^{(1)} & a_0^{(2)}\end{pmatrix}\begin{pmatrix}\boldsymbol{v}_{+0}\\ \boldsymbol{v}_{-0}\end{pmatrix}. \tag{5.11}$$

The boldface letters are vectors with respect to the level number n. Though we have omitted the transpose symbol for the vectors in (5.10), how they form inner products should be evident.

Each term in the exponent of (5.10) is reduced to $(A_0)^2=\sum_{r=1}^{3}(a_0^{(r)})^2$ times a quantity consisting of the Neumann coefficients (their calculations are summarized in appendix C). Using the on-shell condition, $(A_0)^2=-6m_{\mathrm{t}}^2$, and substituting $\mathcal{N}_{\mathrm{t}}$ of (5.9), the square of g_{o} is given by

$$\begin{aligned}g_{\mathrm{o}}^2=\frac{1}{KG^3}&\Bigl(\det(1-T^2)^{-3}\det(1-S\mathcal{V}_3)^2\Bigr)^{-13}\times\\ &\times\det(1-\widetilde{T}^2)^{-3}\det(1-\widetilde{S}\widetilde{\mathcal{V}}_3)^2\,\exp\bigl(6Hm_{\mathrm{t}}^2\bigr)\,,\end{aligned} \tag{5.12}$$

where H has the following expression similar to G (4.24):

$$H=\boldsymbol{v}_{+0}H_{++}\boldsymbol{v}_{+0}+\boldsymbol{v}_{+0}H_{+-}\boldsymbol{v}_{-0}\,, \tag{5.13}$$

with

$$\begin{aligned}H_{++}&=3\,M_0\left[(1-M_0)(1+3M_0)^2\right]^{-1}\bigl(1+4M_0-3M_0^2+2\,M_0T\bigr)\,,\\ H_{+-}&=-\frac{3}{2}\,(1+M_0)\left[(1-M_0)(1+3M_0)^2\right]^{-1}\bigl(1+4M_0-3M_0^2+2\,M_0T\bigr)\end{aligned} \tag{5.14}$$

Note that the three-tachyon coupling calculated here does not take a form which vanishes when the external momenta are put on-shell. This implies that the tachyon wave function Φ_{t} constructed in section 4 is not $\mathcal{Q}_{\mathrm{B}}$-exact.

5.4 $\mathcal{E}_{\mathrm{c}}/T_{25}$

Now we are ready to consider the ratio $\mathcal{E}_{\mathrm{c}}/T_{25}$. The energy density $\mathcal{E}_{\mathrm{c}}$ is given by (5.4) and the D25-brane tension by $T_{25}=1/(2\pi^2g_{\mathrm{o}}^2)$ with g_{o}^2 (5.12). First, we see that the determinant factors cancel between the two since we have

$$\det(1-T^2)^{-3}\det(1-S\mathcal{V}_3)^2=\det\left[(1-M_0)^{3/2}(1+3M_0)^{1/2}\right], \tag{5.15}$$

L	H	$\mathcal{E}_c/T_{25}$
10	-0.0418	12.69
50	-0.1160	19.81
100	-0.1386	22.69
150	-0.1500	24.29
200	-0.1573	25.39
250	-0.1627	26.22
300	-0.1668	26.88

Table 2: H (5.13) and $\mathcal{E}_c/T_{25}$ (5.19) for various values of the cutoff L.

and the corresponding one for the tilded matrices. To prove this, first note the following expression for the 3×3 part of the determinant $\det(1-S\mathcal{V}_3)$:

$$\begin{aligned}\det_{3\times 3}(1-S\mathcal{V}_3) &= 1-3M_0T+3\left(M_0^2-M_+M_-\right)T^2- \\ &\quad -\left(M_+^3+M_-^3+M_0^3-3M_0M_+M_-\right)T^3 \\ &= 1-3\,M_0+3\,T-T^3\,, \end{aligned}\tag{5.16}$$

where we have used (2.15)–(2.17) and (3.18) in obtaining the last expression. Eq. (5.15) is a consequence of (5.16) and an identity for T of (3.19):

$$\left(1-T^2\right)^{-3}\left(1-3M_0+3T-T^3\right)^2=(1-M_0)^{3/2}(1+3M_0)^{1/2}\,. \tag{5.17}$$

Therefore, the ratio between the energy density $\mathcal{E}_c$ and the D25-brane tension T_{25} reads

$$\frac{\mathcal{E}_c}{T_{25}}=\frac{\pi^2}{3G^3}\exp\left(6\,m_t^2\,H\right). \tag{5.18}$$

This ratio is given in terms of G (4.24), the tachyon mass squared m_t^2 determined by (4.27), and H (5.13). The constant K multiplying the VSFT action (2.1) has been cancelled out in (5.18) as of course it should be. If we admit that the tachyon mass is correctly reproduced, namely, $m_t^2=-1$ and $G=\ln 2$ (otherwise the present analysis does not make sense), (5.18) is rewritten as

$$\frac{\mathcal{E}_c}{T_{25}}=\frac{\pi^2}{3\,(\ln 2)^3}\,e^{-6\,H}\,. \tag{5.19}$$

If our classical solution Ψ_c in VSFT represents the perturbative open string vacuum, we should have $\mathcal{E}_c/T_{25}=1$. Namely, the desired value of H is $H=(1/6)\ln(\pi^2/(3(\ln 2)^3)\simeq 0.3817$.

We have carried out numerical analysis of H (5.13) by cutting off the size of the matrices to $L\times L$. The result of our calculation given in table 2 is very different from our expectation. As the cutoff L is increased, $\mathcal{E}_c/T_{25}$ becomes larger far beyond the desired value of one.

The most naive and hasty interpretation of this unwanted behavior of $\mathcal{E}_c/T_{25}$ is that our solution Ψ_c of VSFT does not correspond to the perturbative open string vacuum. However, in the next subsection we shall argue that there are in fact ambiguities in the calculation of $\mathcal{E}_c/T_{25}$.

5.5 Reexamination of $\mathcal{E}_c/T_{25}$

Recall that, though each of $\mathcal{E}_c$ and T_{25} contains determinant factors, they are cancelled out in the ratio (5.18). However, numerical analysis shows that these determinant factors themselves are not finite quantities, and this fact suggests that there might be subtle points in the treatment of the determinants in $\mathcal{E}_c/T_{25}$. Therefore, we shall reexamine the ratio $\mathcal{E}_c/T_{25}$ by adopting different treatments than those in the previous subsections: We do not use the equation of motion for Ψ_c, $\mathcal{Q}\Psi_c + \Psi_c * \Psi_c = 0$, in calculating $\mathcal{E}_c$, nor do we use the commutativity among M_α and various identities originating from (2.15) and (2.16) to simplify the determinants

Without using the equation of motion, the action of our classical solution $\mathcal{S}[\Psi_c]$ is given instead of (5.1) by

$$\mathcal{S}[\Psi_c] = -K\Bigg(\frac{1}{2}\mathcal{N}_c^2\left[\det(1-S^2)\right]^{-13}\det(1-\widetilde{S}^2) + \\ + \frac{1}{3}\mathcal{N}_c^3\left[\det(1-S\mathcal{V}_3)\right]^{-13}\det(1-\widetilde{S}\widetilde{\mathcal{V}}_3)\Bigg)V_{26}\,. \tag{5.20}$$

Then, using (3.11) for $\mathcal{N}_c$ and (5.12) for g_o^2, the ratio $\mathcal{E}_c/T_{25}$ is given instead of (5.18) by

$$\frac{\mathcal{E}_c}{T_{25}} = \frac{\pi^2}{3G^3}\,e^{6m_t^2 H}\,Z\,, \tag{5.21}$$

where the extra factor Z is

$$Z = 3\left(\frac{(Z_{\rm matt})^{13}}{Z_{\rm gh}}\right)^2 - 2\left(\frac{(Z_{\rm matt})^{13}}{Z_{\rm gh}}\right)^3\,, \tag{5.22}$$

with

$$Z_{\rm matt} = \frac{\det(1-S^2)\det(1-S\mathcal{V})}{\det(1-S\mathcal{V}_3)}, \tag{5.23}$$

$$Z_{\rm gh} = \frac{\det(1-\widetilde{S}^2)\det(1-\widetilde{S}\widetilde{\mathcal{V}})}{\det(1-\widetilde{S}\widetilde{\mathcal{V}}_3)}. \tag{5.24}$$

If we use the commutativity among M_α and their identities given in section 2, we have $Z_{\rm matt} = Z_{\rm gh} = 1$ and the previous expression (5.18) is recovered.

Table 3 shows the result of our numerical calculation of $Z_{\rm matt}$, $Z_{\rm gh}$ and Z using level truncation. We did not use the commutativity to reduce $2L \times 2L$ determinant

L	$Z_{\rm matt}$	$Z_{\rm gh}$	Z
2	1.0195	1.0487	0.8253
4	1.0318	1.0734	0.3912
6	1.0396	1.0881	−0.1000
8	1.0450	1.0984	−0.5903
10	1.0492	1.1062	−1.0644
20	1.0615	1.1291	−3.1475
30	1.0683	1.1416	−4.8626

Table 3: $Z_{\rm matt}$ (5.23), $Z_{\rm gh}$ (5.24) and Z (5.22) for various values of the cutoff L.

$\det(1-S\mathcal{V})$ and $3L\times 3L$ one $\det(1-S\mathcal{V}_3)$ to $L\times L$ determinants as we did before. We treated them as they stand (note that both the commutativity and the non-linear identity (2.16) are violated for a finite cutoff L). We used (3.19) as the matrix $S=CT$ in the determinants. As seen from table 3, both $Z_{\rm matt}$ and $Z_{\rm gh}$ gradually deviate from one rather than approach it as the cutoff L is increased. The total factor Z appearing in the ratio $\mathcal{E}_{\rm c}/T_{25}$ deviates from one much faster due mainly to the large power $(Z_{\rm matt})^{39}$ in the last term of (5.22), and it even becomes negative for larger L.

This result itself does not directly remedy the unwanted behavior of $\mathcal{E}_{\rm c}/T_{25}$ given in table 2 but rather worsens it. We even do not know which of the two, $Z=1$ as in section 5.4 or a non-trivial Z here, is a "correct" one. A similar problem may exist also in H. In any case, a lesson we learn from the present analysis is that the ratio $\mathcal{E}_{\rm c}/T_{25}$ is a rather ambiguous quantity. To give a final answer to the potential height problem for our classical solution, we have to understand a basic principle which fixes the ambiguities.

6. Conclusion

Our analysis for the VSFT solution $\Psi_{\rm c}$ given in this paper is still incomplete for drawing a conclusion on whether $\Psi_{\rm c}$ represents the perturbative open string vacuum. Affirmative results are the tachyon and the vector masses (the former was calculated numerically to give an almost expected value, while the latter is exactly zero), and the tachyon-tachyon-vector coupling. However, we got a disappointing result for the potential height $\mathcal{E}_{\rm c}/T_{25}$.

Our remaining problems are now evident. First, we have to clarify the full structure of the fluctuation modes satisfying the wave equation $\mathcal{Q}_{\rm B}\Phi=0$ besides tachyon and massless vector modes. It is also necessary to resolve the problem in constructing the massless vector wave function described in section 4.4. The most important and probably a difficult problem is a reconsideration of the potential height. As discussed in section 5.5, there seems to be ambiguities in the analysis of $\mathcal{E}_{\rm c}/T_{25}$, which we have to settle for giving the final answer. Although our VSFT classical solution $\Psi_{\rm c}$ is given in the Siegel gauge, it is of interest to consider solutions in other gauges. There might be gauges where various quantities appearing in $\mathcal{E}_{\rm c}/T_{25}$ become less singular.

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A. Useful formulas

We quote two basic formulas frequently used in the text for calculating inner products between squeezed states. For any bosonic (fermionic) oscillators satisfying the commutation (anti-commutation) relations,

$$[a_i, a_j^\dagger] = \delta_{ij}\,, \qquad \{c_i, b_j^\dagger\} = \{b_i, c_j^\dagger\} = \delta_{ij}\,, \tag{A.1}$$

and for the Fock vacuum $|0\rangle$ annihilated by (a_i, b_i, c_i) for any i, we have

$$\begin{aligned}
&\langle 0|\exp\left(-\frac{1}{2}a_i A_{ij} a_j - K_i a_i\right)\exp\left(-\frac{1}{2}a_i^\dagger B_{ij} a_j^\dagger - J_i a_i^\dagger\right)|0\rangle = \\
&\qquad = \left[\det\left(1-AB\right)\right]^{-1/2}\exp\left(-\frac{1}{2}JPAJ - \frac{1}{2}KBPK + JPK\right), \qquad \text{(A.2)}\\
&\langle 0|\exp\left(-c_i F_{ij} b_j\right)\exp\left(c_i^\dagger G_{ij} b_j^\dagger + \eta_i b_i^\dagger + c_i^\dagger \zeta_i\right)|0\rangle = \\
&\qquad = \det\left(1-FG\right)\exp\Bigl(\eta(1-FG)^{-1}F\zeta\Bigr)\,, \qquad \text{(A.3)}
\end{aligned}$$

where P in (A.2) is $P = (1-AB)^{-1}$, and η_i and ζ_i in (A.3) are Grassmann-odd variables.

B. Derivation of G (4.24)

In this appendix, we summarize the derivation of the expression (4.24) for G from the original one (4.23). First, each term on the RHS of (4.23) is calculated to give the following form not containing $M_\pm$:

$$
\begin{aligned}
&(\boldsymbol{v}_{-0}-\boldsymbol{v}_{+0},\boldsymbol{v}_{+0})(1-T\mathcal{M})^{-1}T\begin{pmatrix}\boldsymbol{v}_{+0}-\boldsymbol{v}_{-0}\\ \boldsymbol{v}_{-0}\end{pmatrix}=\\
&\quad=-(\boldsymbol{v}_{+0}-\boldsymbol{v}_{-0})D^{-1}T(\boldsymbol{v}_{+0}-\boldsymbol{v}_{-0})+\boldsymbol{v}_{+0}D^{-1}T(1-T)\boldsymbol{v}_{-0}\,, && \text{(B.1)}\\
&2\,(\boldsymbol{v}_{-0}-\boldsymbol{v}_{+0},\boldsymbol{v}_{+0})(1-T\mathcal{M})^{-1}\begin{pmatrix}0\\ \boldsymbol{t}\end{pmatrix}=\\
&\quad=2\,\boldsymbol{v}_{+0}(1-T)D^{-1}\boldsymbol{t}=-2\,\boldsymbol{v}_{+0}D^{-2}(1-T)^3\,(\boldsymbol{v}_{+0}+\boldsymbol{v}_{-0})\,, && \text{(B.2)}\\
&(0,\boldsymbol{t})\mathcal{M}(1-T\mathcal{M})^{-1}\begin{pmatrix}0\\ \boldsymbol{t}\end{pmatrix}=\\
&\quad=\boldsymbol{t}D^{-1}M_0(1-T)\boldsymbol{t}=(\boldsymbol{v}_{+0}+\boldsymbol{v}_{-0})D^{-3}M_0(1-T)^5(\boldsymbol{v}_{+0}+\boldsymbol{v}_{-0})\,. && \text{(B.3)}
\end{aligned}
$$

In every calculation we have used (4.18) for $(1-T\mathcal{M})^{-1}$ and defined

$$D=(1-M_0)(1+T)\,. \tag{B.4}$$

We also made repeated use of the identities (2.15), (2.24) and (2.25). In particular, at the first equality of (B.2) we have used

$$\boldsymbol{v}_{+0}M_-+\boldsymbol{v}_{-0}M_+=0\,, \tag{B.5}$$

obtained by taking the transpose of (2.24) and using that M_α is a symmetric matrix

$$M_\alpha^{\mathrm{T}}=M_\alpha\,,\qquad(\alpha=0,\pm)\,. \tag{B.6}$$

In obtaining the last expressions of (B.2) and (B.3), we have used (4.22) for $\boldsymbol{t}$.

Then, summing (B.1)–(B.3) and the LHS of (2.28), and using that $\boldsymbol{v}_{-0}\mathcal{O}\boldsymbol{v}_{+0}=\boldsymbol{v}_{+0}\mathcal{O}\boldsymbol{v}_{-0}$ and $\boldsymbol{v}_{-0}\mathcal{O}\boldsymbol{v}_{-0}=\boldsymbol{v}_{+0}\mathcal{O}\boldsymbol{v}_{+0}$ when $\mathcal{O}$ consists solely of M_0 and T commutative with C, we get primitive expressions of G_{++} and G_{+-}. The final expressions (4.25) and (4.26) are obtained by substituting

$$D^{-1}=\left[(1-M_0)(1+3M_0)\right]^{-1}(1+2M_0-M_0T)\,, \tag{B.7}$$

combining all terms over a common denominator, and repeatedly using $M_0T^2=(1+M_0)T-M_0$ (see (3.18)) to reduce the power of T in the numerator.

C. Derivation of $\boldsymbol{H}$ (5.13)

The main task in obtaining the concise expression (5.13) for H is the calculation of the first three terms in the exponent in the last expression of (5.10):

$$-\frac{1}{2}\boldsymbol{V}(1-S\mathcal{V}_3)^{-1}S\boldsymbol{V}+\boldsymbol{V}(1-S\mathcal{V}_3)^{-1}\boldsymbol{t}A_0-\frac{1}{2}A_0\boldsymbol{t}\mathcal{V}_3(1-S\mathcal{V}_3)^{-1}\boldsymbol{t}A_0-\frac{1}{2}V_{00}(A_0)^2\,. \tag{C.1}$$

Let us first evaluate these three terms.

$-(1/2)\boldsymbol{V}(1-S\mathcal{V}_3)^{-1}S\boldsymbol{V}$

First, we have

$$(1-S\mathcal{V}_3)^{-1} = (1-T\mathcal{M}_3)^{-1}$$
$$= D_3^{-1}\begin{pmatrix}(1-M_0)(1+T) & T(M_+ + M_-T) & T(M_- + M_+T)\\ T(M_- + M_+T) & (1-M_0)(1+T) & T(M_+ + M_-T)\\ T(M_+ + M_-T) & T(M_- + M_+T) & (1-M_0)(1+T)\end{pmatrix},\quad\text{(C.2)}$$

where D_3 is the 3×3 determinant of $1-S\mathcal{V}_3$, (5.16):

$$D_3 = 1-3\,M_0+3\,T-T^3\,. \qquad \text{(C.3)}$$

Using this and $\boldsymbol{V}$ in (5.11), the first term of (C.1) is calculated as

$$-\frac{1}{2}\boldsymbol{V}(1-S\mathcal{V}_3)^{-1}S\boldsymbol{V} = -\frac{1}{2}\boldsymbol{V}(1-S\mathcal{V}_3)^{-1}T\,C\boldsymbol{V} =$$
$$= -\frac{1}{2}\left(\boldsymbol{v}_{+0},\boldsymbol{v}_{-0}\right)\begin{pmatrix}a_0^{(2)} & a_0^{(3)} & a_0^{(1)}\\ a_0^{(3)} & a_0^{(1)} & a_0^{(2)}\end{pmatrix}(1-S\mathcal{V}_3)^{-1}\,T\begin{pmatrix}a_0^{(3)} & a_0^{(2)}\\ a_0^{(1)} & a_0^{(3)}\\ a_0^{(2)} & a_0^{(1)}\end{pmatrix}\begin{pmatrix}\boldsymbol{v}_{+0}\\ \boldsymbol{v}_{-0}\end{pmatrix}$$
$$= \frac{1}{4}(A_0)^2\left(\boldsymbol{v}_{+0},\boldsymbol{v}_{-0}\right)D_3^{-1}T\,\times$$
$$\times\begin{pmatrix}(1-T)(1-T+3TM_+) & (1+T)(-2+T)(1-M_0)\\ (1+T)(-2+T)(1-M_0) & (1-T)(1-T+3TM_-)\end{pmatrix}\begin{pmatrix}\boldsymbol{v}_{+0}\\ \boldsymbol{v}_{-0}\end{pmatrix}.\qquad\text{(C.4)}$$

In the last line we have replaced $M_\mp$ in the upper-left/lower-right component of the 2×2 matrix by $1-M_0-M_\pm$ via (2.15). This is because we can remove the remaining $M_\pm$ in the matrix by using (2.25), i.e., $M_\pm\boldsymbol{v}_{\pm0}=\boldsymbol{v}_{\pm0}-M_0\boldsymbol{v}_{\mp0}$. After carrying out this procedure, we finally obtain

$$-\frac{1}{2}\boldsymbol{V}(1-S\mathcal{V}_3)^{-1}S\boldsymbol{V} = \frac{1}{4}(A_0)^2\left(\boldsymbol{v}_{+0},\boldsymbol{v}_{-0}\right)D_3^{-1}T(1-T)\begin{pmatrix}1+2T & -2-T\\ -2-T & 1+2T\end{pmatrix}\begin{pmatrix}\boldsymbol{v}_{+0}\\ \boldsymbol{v}_{-0}\end{pmatrix}.\qquad\text{(C.5)}$$

$\boldsymbol{V}(1-S\mathcal{V}_3)^{-1}\boldsymbol{t}A_0$

Eq. (C.2) together with (4.19), (4.21) and (4.22), i.e.,

$$(1-M_0)(1+T)\boldsymbol{t} = -(1-T)^2\left(\boldsymbol{v}_{+0}+\boldsymbol{v}_{-0}\right).\qquad\text{(C.6)}$$

gives

$$(1-S\mathcal{V}_3)^{-1}\boldsymbol{t}A_0 = -D_3^{-1}(1-T)\,\times$$
$$\times\begin{pmatrix}(1-T)(\boldsymbol{v}_{+0}+\boldsymbol{v}_{-0}) & T(\boldsymbol{v}_{+0}-T\boldsymbol{v}_{-0}) & T(\boldsymbol{v}_{-0}-T\boldsymbol{v}_{+0})\\ T(\boldsymbol{v}_{-0}-T\boldsymbol{v}_{+0}) & (1-T)(\boldsymbol{v}_{+0}+\boldsymbol{v}_{-0}) & T(\boldsymbol{v}_{+0}-T\boldsymbol{v}_{-0})\\ T(\boldsymbol{v}_{+0}-T\boldsymbol{v}_{-0}) & T(\boldsymbol{v}_{-0}-T\boldsymbol{v}_{+0}) & (1-T)(\boldsymbol{v}_{+0}+\boldsymbol{v}_{-0})\end{pmatrix}A_0$$

$$
\begin{aligned}
&= -D_3^{-1}(1-T)\times\\
&\times\begin{pmatrix}(1-T)a_0^{(1)}+Ta_0^{(2)}-T^2a_0^{(3)} & (1-T)a_0^{(1)}+Ta_0^{(3)}-T^2a_0^{(2)}\\ (1-T)a_0^{(2)}+Ta_0^{(3)}-T^2a_0^{(1)} & (1-T)a_0^{(2)}+Ta_0^{(1)}-T^2a_0^{(3)}\\ (1-T)a_0^{(3)}+Ta_0^{(1)}-T^2a_0^{(2)} & (1-T)a_0^{(3)}+Ta_0^{(2)}-T^2a_0^{(1)}\end{pmatrix}\begin{pmatrix}\boldsymbol{v}_{+0}\\ \boldsymbol{v}_{-0}\end{pmatrix}. \qquad (C.7)
\end{aligned}
$$

The second term of (C.1) is easily calculated by using (C.7):

$$
\begin{aligned}
\boldsymbol{V}(1-S\mathcal{V}_3)^{-1}\boldsymbol{t}A_0 = &\frac{1}{2}(A_0)^2\,(\boldsymbol{v}_{+0},\boldsymbol{v}_{-0})D_3^{-1}(1-T)\times\\
&\begin{pmatrix}1-3T-T^2 & 1+2T^2\\ 1+2T^2 & 1-3T-T^2\end{pmatrix}\begin{pmatrix}\boldsymbol{v}_{+0}\\ \boldsymbol{v}_{-0}\end{pmatrix}. \qquad (C.8)
\end{aligned}
$$

<u>$-(1/2)A_0\boldsymbol{t}\mathcal{V}_3(1-S\mathcal{V}_3)^{-1}\boldsymbol{t}A_0$</u>

Let us calculate

$$
-\frac{1}{2}\,A_0\boldsymbol{t}\mathcal{V}_3(1-S\mathcal{V}_3)^{-1}\boldsymbol{t}A_0 = -\frac{1}{2}\,A_0\boldsymbol{t}\mathcal{M}_3(1-S\mathcal{V}_3)^{-1}\boldsymbol{t}A_0\,. \qquad (C.9)
$$

First, using (C.7) we have

$$
\begin{aligned}
-A_0\mathcal{M}_3(1-S\mathcal{V}_3)^{-1}\boldsymbol{t}A_0 &= \frac{1}{2}\,(A_0)^2D_3^{-1}(1-T)\sum_{\pm}\Big((2-3T+T^2)M_0 -\\
&\qquad -(1+2T^2)M_{\pm}+(-1+3T+T^2)M_{\mp}\Big)\boldsymbol{v}_{\pm0}\\
&= -\frac{1}{2}\,(A_0)^2D_3^{-1}(1-2T)(1-T)^2\,(\boldsymbol{v}_{+0}+\boldsymbol{v}_{-0}). \qquad (C.10)
\end{aligned}
$$

In obtaining the last line we have used the same technique as used in passing from (C.4) to (C.5). As $\boldsymbol{t}$ to be multiplied to (C.10) to get (C.9), we use the following expression:

$$
\begin{aligned}
\boldsymbol{t} &= -D^{-1}(1-T)^2(\boldsymbol{v}_{+0}+\boldsymbol{v}_{-0})\\
&= -(1+3M_0)^{-1}(1+T)(\boldsymbol{v}_{+0}+\boldsymbol{v}_{-0})\,, \qquad (C.11)
\end{aligned}
$$

where we have used (B.7) and the identity

$$
(1+2M_0-M_0T)\,(1-T)^2 = (1-M_0)(1+T)\,, \qquad (C.12)
$$

which is obtained by repeated use of (3.18). Then, we have

$$
\begin{aligned}
-\frac{1}{2}\,A_0\boldsymbol{t}\mathcal{V}_3(1-S\mathcal{V}_3)^{-1}\boldsymbol{t}A_0 = &\frac{1}{4}(A_0)^2\,(\boldsymbol{v}_{+0}+\boldsymbol{v}_{-0})(1+3M_0)^{-1}D_3^{-1}\times\\
&\times(1-2T)(1-T)^2(1+T)(\boldsymbol{v}_{+0}+\boldsymbol{v}_{-0})\,. \qquad (C.13)
\end{aligned}
$$

<u>Total of H</u>

Now we have obtained the first three terms in (C.1), and the total of H is given by

$$H = \left(-\frac{1}{2}(A_0)^2\right)^{-1}\Big[(\mathrm{C.5}) + (\mathrm{C.8}) + (\mathrm{C.13})\Big] + V_{00} + \boldsymbol{t}(1+T)^{-1}\boldsymbol{t}\,. \qquad \text{(C.14)}$$

We use (2.28) for V_{00} and

$$\boldsymbol{t}(1+T)^{-1}\boldsymbol{t} = (\boldsymbol{v}_{+0} + \boldsymbol{v}_{-0})(1+3M_0)^{-2}(1+T)(\boldsymbol{v}_{+0} + \boldsymbol{v}_{-0})\,, \qquad \text{(C.15)}$$

obtained from (C.11). Then, using

$$D_3^{-1} = \big[(1-M_0)^2(1+3M_0)\big]^{-1}\big(1 + M_0 - M_0^2 - M_0 T\big)\,, \qquad \text{(C.16)}$$

in (C.5), (C.8) and (C.13), and putting all terms in (C.14) over a common denominator, and reducing the power of T by repeated use of (3.18) as we did for G, we arrive at (5.13).

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